J. Fluid Mech. (1965), *vol.* 22, *part* 4, *pp.* 701–707
Printed in Great Britain

A simple demonstration of the Hartmann layer

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(Received 2 December 1964)

This paper gives the theory of the motion of a finitely conducting, viscous liquid between long, concentric, rotating cylinders under a radial magnetic field in the case of high Hartmann number. Because the current and vorticity contents of a Hartmann layer are related, it is easy to predict the behaviour in both steady and unsteady states. Experiments with mercury which confirm the theory closely are described. As these were developed for an educational film on magnetohydrodynamics, they have the attraction that the results are immediately obvious to the observer.

Introduction

In the course of making an expository film on magnetohydrodynamics (Shercliff 1965) we have developed a new experiment which demonstrates strikingly the main property of the Hartmann layer, namely, that its current content and vorticity content are proportional to one another. The experiment is very convenient because it does not involve a flow loop, pump, etc., and the essential results are apparent to the naked eye.

The experiment is performed in an annular tank of mercury, contained between two concentric, non-conducting cylinders rotated steadily in order to make the fluid move in concentric circles. A radial magnetic field B can be applied externally. Currents are then induced axially (i.e. in the $\mathbf{v} \times \mathbf{B}$ direction), but no current enters or leaves the fluid. The Hartmann number M or $g\bar{B}(\sigma/\eta)^{\frac{1}{2}}$, based on annular gap g and mean field strength $\bar{B}$, is large (> 200). The symbols σ and η denote electrical conductivity and viscosity, respectively. The Hartmann layers at the cylinders are relatively thin (with thickness of order g/M) and can be treated as current/vortex sheets.

Before the experiment and results are described we first consider the theory of the motion at high Hartmann number. This is one of the many cases where the theory of MHD is easier than the corresponding problem in the absence of electromagnetic effects. Moreover, the experiment itself is very much better behaved with the magnetic field than without it because the magnetic field suppresses unwanted secondary flows.

Theory: Steady state

Consider first the problem of axisymmetric steady flow between concentric, non-conducting cylinders of such length that end effects can be neglected. Let there be a purely radial magnetic field imposed. One or both of the cylinders rotates steadily and the resulting fluid velocity is assumed to be everywhere azimuthal, i.e. no secondary flow or instability occurs. This assumption is borne out in the experiments. Conditions are invariant in the axial direction and all currents are purely axial. The specification of the problem is completed by the condition that the currents do not enter or leave the fluid, but close on themselves at the remote ends.

The problem is an easy exercise in linear magnetohydrodynamics and the velocity is given by an expression in powers of r, the radius. Here we propose to go straight to the asymptotic form at high M, which is much more instructive.

When M is large one can apply the usual technique (see, for example, Shercliff 1956) of neglecting viscosity in the bulk of the flow and using the standard boundary condition for the Hartmann current/vortex sheets at the walls

$$J = (\sigma\eta)^{\frac{1}{2}}\Delta v, \tag{1}$$

where J is the current content of a sheet, per unit perimeter, and Δv is its vorticity content per unit perimeter, i.e. the jump in tangential velocity across it. This makes an interesting contrast with ordinary inviscid hydrodynamics where Δv, the slip at a wall, is unconstrained. In the magnetic case, however small the viscosity is, it still can govern the problem via the boundary condition (1). Note that B itself does not appear explicitly in (1) although B controls the problem, through M.

In the bulk of the flow, the azimuthal component of the equation of inviscid motion is very degenerate; there is no azimuthal component of acceleration and no azimuthal pressure gradient. So the magnetic force $\mathbf{j} \times \mathbf{B}$ (which would be azimuthal) must vanish and there can be no currents $\mathbf{j}$ in the bulk of the flow. The fluid moves in such a way that $\mathbf{v} \times \mathbf{B}$ is irrotational and can be balanced by the axial electrostatic field $\mathbf{E}$. This implies that the fluid moves like a solid body, a result which can be seen in various ways, for example:

(i) only then can any loop drawn in the fluid link a constant magnetic flux and thereby avoid inducing eddy currents in the bulk of the fluid;

(ii) since curl $\mathbf{E} = 0$ in the steady state, the axial electric field is uniform and, in the absence of currents, $\mathbf{v} \times \mathbf{B}$ must be uniform. But $\mathbf{B}$ is proportional to $1/r$ and so $\mathbf{v}$ must be proportional to r, and we have solid-body rotation.

Except in the special case where the two cylinders have the same angular velocity, the solid-body rotation of the fluid involves slip at both cylinders. It has to be *both* cylinders because *slip* implies *current* and the current up one Hartmann sheet can only return down the other, there being no current in the bulk of the flow or out of the fluid. This simple condition suffices to determine the two slips and the value of the fluid's uniform angular velocity.

Let the angular velocities of the two cylinders be ω_1 and ω_2 and their radii r_1 and r_2 (see figure 1). If the angular velocity of the fluid is ω, the two slips are

$$\Delta v_1 = r_1(\omega - \omega_1), \quad \Delta v_2 = r_2(\omega_2 - \omega). \tag{2}$$

The conservation of current flow demands that

$$2\pi r_1(\sigma\eta)^{\frac{1}{2}}\Delta v_1 = 2\pi r_2(\sigma\eta)^{\frac{1}{2}}\Delta v_2, \quad \text{i.e. } r_1\Delta v_1 = r_2\Delta v_2. \tag{3}$$

This is also the condition that the viscous torques on the two cylinders are equal, since the viscous shear stress at a wall $= B\Delta v(\sigma\eta)^{\frac{1}{2}}$, and $B \propto 1/r$.

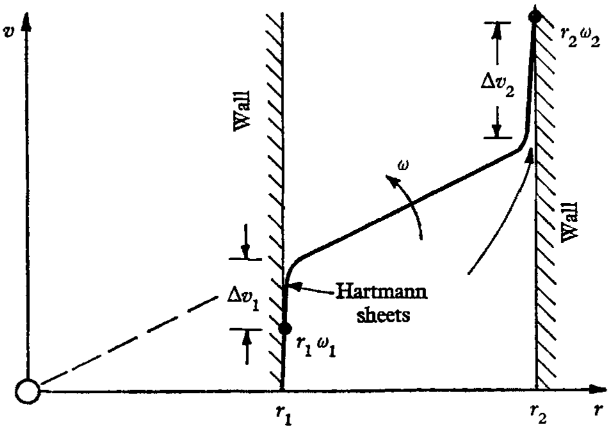

FIGURE 1. The velocity profile between rotating cylinders at large Hartmann number.

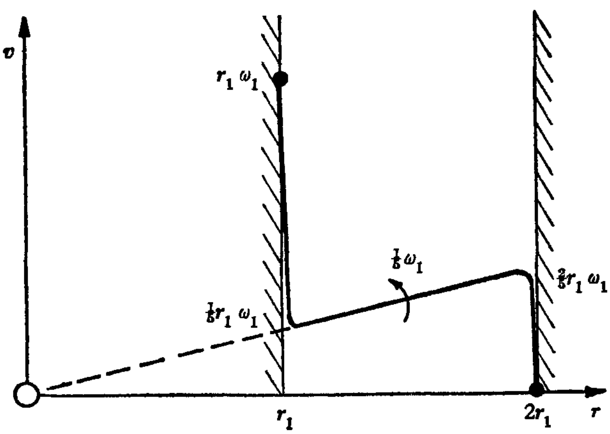

FIGURE 2. The velocity profile for the case corresponding to the experiment.

From (2) and (3) it follows that

$$\omega = (r_1^2\omega_1 + r_2^2\omega_2)/(r_1^2 + r_2^2). \tag{4}$$

In the experiment reported here, $\omega_2 = 0$ and $r_2 = 2r_1$, and one should expect the result

$$\omega = \tfrac{1}{5}\omega_1, \tag{5}$$

at least for cylinders of great length. The corresponding velocity profile is shown in figure 2.

Note that the induced magnetic field due to the currents is azimuthal, i.e. parallel to the velocity, and does not affect the problem. If the imposed magnetic field is not purely radial, although still axisymmetric, the results are not affected. Again $\text{curl}\,\mathbf{v}\times\mathbf{B} = 0$ implies solid-body rotation in the bulk of the fluid. The argument in terms of a loop drawn in the fluid makes this self-evident. This fact means that extreme care in avoiding fringing of the magnetic field in an experiment is not necessary.

Of more concern in an experiment is the question whether the fact that the fluid must be of finite axial length has serious repercussions. One effect of the ends is to interrupt the axial current flows. From the loose analogy with flow in rectangular channels under transverse fields (Shercliff 1953) one is encouraged to expect that in an experiment the currents up and down the Hartmann sheets will find their way radially across the ends of the fluid in some kind of thin layer probably of thickness of order $g/M^{\frac{1}{2}}$. In our experiment one end is a horizontal free surface, the other a solid wall, attached to the outer cylinder, the axial length being approximately twice the gap dimension.

Any departure of the imposed magnetic field from being radial will affect these end layers because the currents there have to flow nearly parallel to the field to avoid producing excessive $\mathbf{j}\times\mathbf{B}$ forces.

The other major effect of the ends is that the velocity ceases to be independent of axial distance there and so the centrifugal force distribution $\rho(\mathbf{v}.\text{grad})\mathbf{v}$ (where ρ is density) is rotational and cannot be balanced by pressure. There will thus be a tendency towards secondary flow involving azimuthal vorticity. The azimuthal-induced fields $\mathbf{B}_i$ are very small in practice, but they too would tend to promote secondary flow because of the part $(\mathbf{B}_i.\text{grad})\,\mathbf{B}_i/\mu$ of $\mathbf{j}\times\mathbf{B}$, which acts radially and is rotational at the ends.

Any secondary flow would involve axial velocities, however, and these tend to be directly damped by the radial magnetic field, because azimuthal currents can circulate freely. The evidence of the experiments is that secondary flow is not a serious problem when the magnetic field is on.

These end-effect problems await analysis. An obvious first step would be to make the analysis linear by ignoring secondary flow.

Theory: Unsteady state

The transient behaviour when the state of rotation of the cylinders is suddenly changed is next analysed approximately for the case of M large. Again we assume that the flow is axisymmetric, with only azimuthal velocities, a radial imposed field and axial induced currents, the remote ends precluding current flow out of the fluid.

First, consider the transient behaviour of departures from solid-body rotation in the bulk of the flow where viscosity is ineffectual. The relaxation time is of order $\rho/\sigma B^2$, which is of order M times shorter than the overall relaxation time (6) which emerges from the analysis of viscous wall effects below. Evidently departures from solid-body rotation die out very swiftly, if they occur at all. In watching the experiment one is very forcibly struck by the contrast between the strength of the magnetic 'rigidity' of the bulk of the fluid and the relative feebleness of the viscous coupling to the walls.

So it is a good approximation to treat the bulk of the flow as undergoing solid-body rotation at an angular velocity ω, which can vary with time.

The equation of motion for the fluid can be written down in terms of its total rate of change of moment of momentum. The torque on it about the axis can be calculated from the viscous shear stresses $B\Delta v(\sigma\eta)^{\frac{1}{2}}$ at the walls, there being no net magnetic torque on the whole fluid if no current leaves the fluid, for

$$\text{magnetic torque} = \int_{r_1}^{r_2} 2\pi r^2 jB\,dr = Br\int_{r_1}^{r_2} 2\pi rj\,dr = 0,$$

Br being constant. Hence we have, per unit axial length,

$$\begin{aligned}\text{viscous torque} &= 2\pi r_2^2 B_2 \Delta v_2(\sigma\eta)^{\frac{1}{2}} - 2\pi r_1^2 B_1 \Delta v_1(\sigma\eta)^{\frac{1}{2}} \\ &= 2\pi Br(\sigma\eta)^{\frac{1}{2}}\,(r_2^2\omega_2 + r_1^2\omega_1 - \omega(r_2^2+r_1^2)), \text{ by (2)} \\ &= \tfrac{1}{2}\pi\rho(r_2^4 - r_1^4)\,(d\omega/dt),\end{aligned}$$

using the moment of inertia of the fluid and ignoring the inertia of the Hartmann sheets. This simple differential equation shows that the steady value of ω already discussed is approached exponentially from above or below with a characteristic time

$$\rho(r_2^2 - r_1^2)/4Br(\sigma\eta)^{\frac{1}{2}}. \tag{6}$$

Experiments

Figure 3 shows in section the apparatus in which the experiments were performed. It was symmetric about the vertical axis XX. The magnet poles PP were connected by a yoke (not shown) which linked an exciting winding. In the experiments the radial field was approximately 0·20 Wb/m^2 at a radius of 6·0 cm. Mercury M was contained in the annular tank formed by the poles and the non-conducting base B. The inner pole was covered by an inverted cylinder C which could be rotated at about one revolution per second about the axis by means of the shaft S, driven by a motor below. This cylinder and both pole faces were given an insulating coating. Mercury in the narrow space between the cylinder and the inner pole would have no effect on the experiment. Details of bearings are omitted from figure 3.

The state of motion of the mercury was revealed by the tethered, slender float F, mounted in bearings on the light, pivoted, counterbalanced arm AA which constrained it to move on a circular path, concentric with the cylinders. The float had the right buoyancy to enable the paddles GG attached to it to remain in the centre of the mercury. There were four paddles, arranged in the form of a cross when seen from above, for the purpose of revealing the vorticity of the mercury. The paddles were non-conducting, but would not impede the currents which would be vertical. (Note that conducting paddles could suffer unwanted, induced $\mathbf{j}\times\mathbf{B}$ forces.) No material used in the float had magnetic properties. The disk D on top of the float and the top of the inner cylinder were painted with a simple black and white pattern to make their rotation clearer in the film.

Various experiments were performed. With the field switched off, the motion was rather ill-behaved because of secondary flows and instability. Nevertheless, it was possible to achieve some fairly steady motions in which, with the inner cylinder rotating clockwise, the vorticity float rotated anticlockwise while orbiting clockwise round the axis at about half the speed of the inner cylinder, just as would be expected if the velocity fell slowly from a maximum at the inner cylinder to zero at the outer one, in a steady, viscous flow.

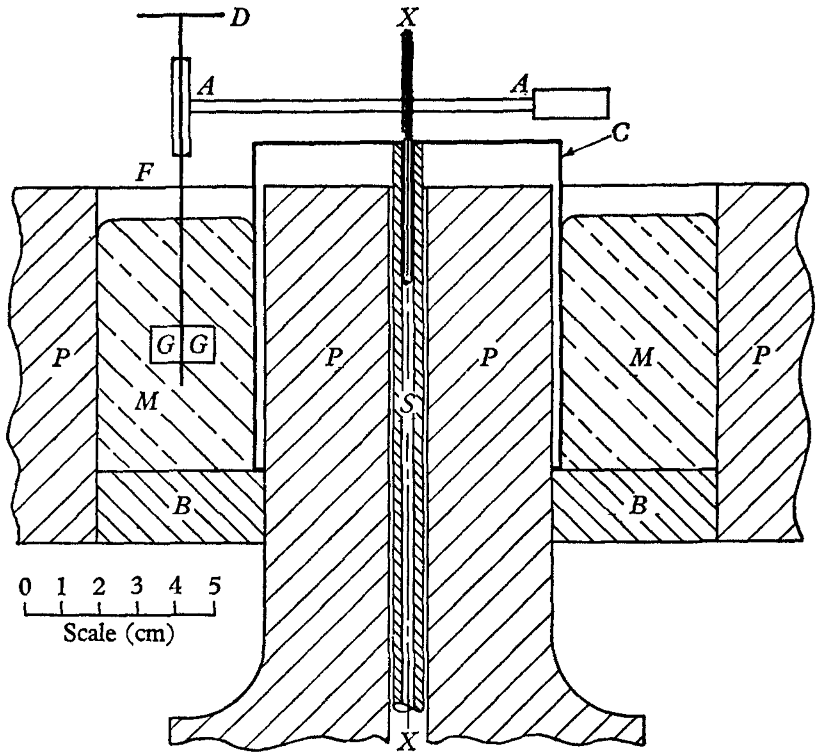

FIGURE 3. Section of apparatus.

The changes which occurred as soon as the field was turned on were dramatic to behold, though fully expected. They included:

(i) the total disappearance of unsteadiness, etc., due to secondary motions,

(ii) the immediate reversal of the vorticity revealed by the float, and

(iii) the immediate onset of solid-body rotation in the mercury, with slip at both cylinders. The angular velocities of the float itself and of its orbit round the axis became identical and equal to one-fifth of the angular velocity of the inner cylinder, in accordance with the prediction (5) to a high order of accuracy. This ratio was apparent to the eye simply by counting revolutions.

Experiments were also conducted where the fluid motion was started from rest with the field on all the time. The inner cylinder would start to rotate at full speed virtually instantaneously, but the fluid approached its steady motion much more slowly. The vorticity float indicated that the fluid accelerated exactly like a solid body. The striking contrast between the magnetic 'rigidity' of the mercury and the feeble viscous coupling to the walls has already been referred to.

From a scrutiny of successive frames of the film record of the starting process, the departure of the angular velocity of the fluid from its final steady value was found to decay in an exponential fashion with a characteristic 1 : *e* time of approximately 30 sec. This value should be compared with the prediction (6) for a system of infinite length which gives a value of 34 sec for the characteristic time if the approximate values $B = 0{\cdot}20\,\mathrm{Wb/m^2}$ at $r = 0{\cdot}060$ m, $r_2 = 2r_1 = 0{\cdot}080$ m, $\rho = 13{,}600\,\mathrm{kg/m^3}$, $\sigma = 1{\cdot}05 \times 10^6$ mho/m, $\eta = 1{\cdot}55 \times 10^{-3}$ kg/m.sec, are inserted. The observed time is slightly smaller than that predicted, probably because B was measured near the top and may have been stronger over much of the gap.

When the cylinder was suddenly stopped after steady rotation had been established, the field being kept on, the decay of the motion was according to a similar exponential law, but the 1 : *e* time was 20 sec, not 30 sec. One may speculate that this difference was due to viscous drag on the stationary base.

Concluding remarks

We have described an MHD demonstration experiment which, without the need for particular care over the uniformity of the magnetic field, etc., behaves in a very docile manner with none of the unsteadiness or other complications which plague so many ordinary fluid-mechanical demonstrations. Moreover, the results are immediately intelligible in terms of simple ideas such as the properties of the Hartmann layer. The most difficult part of the apparatus to make is the vorticity float and its carriage. We wish particularly to acknowledge the skill and patience of Jim Henry, who constructed the apparatus. The work was supported by the National Science Foundation.

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